

UV contribution and power dependence on Λ_{QCD} of Adler function

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Abstract

We formulate a way to separate UV and IR contributions to the Adler function and discuss how $\Lambda_{\text{QCD}}^2/Q^2$ dependence is encoded in the UV contribution within perturbative QCD.

Keywords: QCD, Summation of perturbation theory

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Perturbative QCD has made remarkable progress in recent years. Thanks to developments in computational technology, the first few to several terms of perturbative series have become available for a number of physical quantities. Due to severe infrared (IR) divergences inherent in perturbative QCD, it has become a standard procedure in many of these computations to factorize ultraviolet (UV) and IR contributions [1]. As more accurate predictions became available, it is also becoming practically important to factorize IR renormalons, in addition to IR divergences. An IR renormalon reflects the IR structure of an observable in terms of perturbative QCD and induces a diverging behavior of the perturbative series. For observables which permit operator product expansion (OPE), factorization can be carried out more systematically. In this case, an IR renormalon in a Wilson coefficient induces a perturbative uncertainty of the same order of magnitude as the associated nonperturbative matrix element. This makes it necessary to subtract the IR renormalon from the perturbative evaluation of the Wilson coefficient and to absorb it into the matrix element, which also agrees with the concept of Wilsonian approach. So far this procedure has not been formulated completely, and such a formulation is requisite for precision analyses of QCD in near future.

The Adler function is defined from the derivative of the hadronic vacuum polarization of the photon. It was originally introduced for a phenomenological analysis of the muon anomalous magnetic moment and the spectra of muonic atoms [2]. Since then, it has been playing an important role in precise calculations of the muon anomalous magnetic moment, running of the QED coupling constant $\alpha_{\text{QED}}(k)$ from $k = 0$ to M_Z [3], R ratio in e^+e^- collision [4], the inclusive τ lepton hadronic decay [5], etc. Furthermore, the Adler function serves as an ideal laboratory for various theoretical tests. For instance, dispersion relation, sum rules, lattice QCD calculations, perturbative QCD predictions, renormalons, various models of IR physics, predictions of supersymmetric QCD, etc., have been examined.

According to the analysis of IR renormalons, the perturbative series of the Adler function contains a renormalon which induces an order $\Lambda_{\text{QCD}}^4/Q^4$ uncertainty. That is, it has

the same dimension as the leading nonperturbative matrix element given by the local gluon condensate [6]. Recently perturbative series of the plaquette on lattice, which is similar to the Adler function in the continuum limit, has been computed up to 35 loops and a renormalon behavior of order $\Lambda_{\text{QCD}}^4/Q^4$ was observed [7]. It has the same order of magnitude as the gluon condensate and the observation supports our understanding that IR renormalons appear with the same dimensions as the nonperturbative matrix elements. However, this does not mean that we understand all the power corrections $\sim (\Lambda_{\text{QCD}}/Q)^n$ [8]. There may be power corrections which originate from UV contributions. In OPE Wilson coefficients may contain power corrections. To predict each Wilson coefficient accurately it is important to subtract IR renormalons from the perturbative series of the Wilson coefficient. This concurrently defines the associated nonperturbative matrix element accurately.

In this Letter we formulate a method to extract UV contributions to the Adler function, which can be used in OPE.¹ Related subjects have been studied in [6, 9, 10] (see also [16, 17]), in which UV and IR contributions have been separated and their nature has been elucidated. It was shown that IR contributions induce order $\Lambda_{\text{QCD}}^4/Q^4$ renormalon uncertainty to the perturbative prediction and an explicit integral representation has been given for the UV contribution [18]. Existence of a $\Lambda_{\text{QCD}}^2/Q^2$ dependence in the UV contribution has been discussed, e.g., using a resummation of the perturbative series [18], and in certain model calculations [16, 17]. Our work can be regarded as an extension of the analyses in refs. [9, 10, 18]. We study the (reduced) Adler function D_{β_0} with an explicit IR cut-off μ_f [10], and in the large- β_0 approximation [9]. It gives a natural definition of the Wilson coefficient of the Adler function based on the Wilsonian picture. We show that there exists a genuine UV part, which satisfies $D_{\text{UV}} = D_{\beta_0}(\mu_f) + \mathcal{O}(\mu_f^4/Q^4)$ and is independent of μ_f . Furthermore, D_{UV} can be expressed as a sum of a logarithmic term² and a $\Lambda_{\text{QCD}}^2/Q^2$ term. We also discuss its scheme dependence. We believe that these add information to our previous knowledge, and moreover, the formulation provides a simple and clear picture which would be useful in accurate OPE analyses.

We adapt a formulation used in the analysis of the static QCD potential [11, 12, 14] after appropriate modifications. In the case of the static potential a “Coulomb+linear” form (with logarithmic correction at short-distances) is extracted as the UV contribution, which reproduces lattice results at $r \lesssim 0.25$ fm. This feature provides a guide to our analysis of the UV contribution to the Adler function, in particular concerning power dependence on Λ_{QCD}/Q .

We define the reduced Adler function by

$$D(Q^2) = 4\pi^2 Q^2 \frac{d\Pi(-Q^2)}{dQ^2} - 1, \quad Q^2 = -q^2. \quad (1)$$

[The Adler function is given by $1 + D(Q^2)$ up to a convention dependent normalization

¹ In conventional analyses of renormalons, a UV scale is assumed to be much larger than any scale involved in the calculation. In this Letter, however, we use the terminology “UV” for scales above the factorization scale μ_f in the context of OPE. In particular Q is regarded as a UV scale.

² By a “logarithmic term” we mean a term which is closest to $(Q^2/\Lambda_{\text{QCD}}^2)^P$ with $P = 0$ in the entire range $0 < Q^2 < \infty$, if it is compared with a single power dependence on Q^2 (for an integer P); see eq. (18) and Fig. 2.

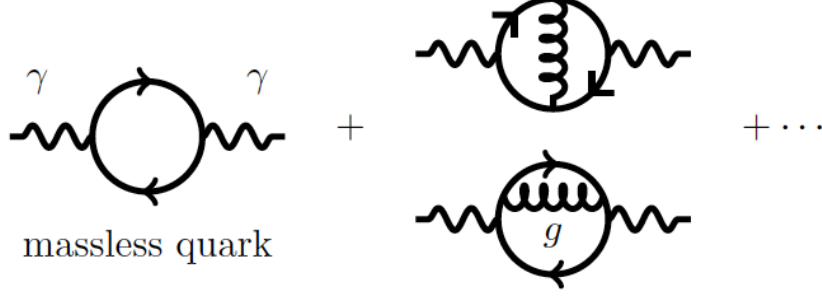


Figure 1: Feynman diagrams for $\Pi(q^2)$.

factor.] $\Pi(q^2)$ denotes the hadronic vacuum polarization given by

$$i \int d^4x e^{iq \cdot x} \langle 0 | T J^\mu(x) J^\nu(0) | 0 \rangle = (q^\mu q^\nu - g^{\mu\nu} q^2) \Pi(q^2), \quad (2)$$

in terms of the correlator of the quark current operator $J^\mu(x) = \bar{q}(x) \gamma^\mu q(x)$. For simplicity we consider one massless quark flavor only. We examine $D(Q^2)$ in the deep Euclidean region $Q^2 \gg \Lambda_{\text{QCD}}^2$.

In perturbative QCD, series expansion of an observable in the strong coupling constant α_s is expected to be an asymptotic series. An IR renormalon is a singularity of the Borel transform of a perturbative series on the positive real axis in the complex Borel plane. The singularity closest to the origin gives rise to the leading asymptotic behavior of the perturbative series. In the case of $D(Q^2)$, the term of the asymptotic series becomes minimal at order $n_* \approx 8\pi/(\beta_0 \alpha_s)$ (β_0 is the coefficient of the one-loop beta function of α_s) and grows rapidly beyond that order. Truncation of the asymptotic series at order n_* induces a theoretical uncertainty of order $(\Lambda_{\text{QCD}}/Q)^4$. On the other hand, using OPE of the current correlator for large Q^2 , the reduced Adler function can be expressed by the vacuum expectation values (VEVs) of gauge and Lorentz invariant local operators as

$$D(Q^2) = d_1 + d_{GG} \frac{\langle 0 | G^{\mu\nu a} G_{\mu\nu}^a | 0 \rangle}{Q^4} + \dots, \quad (3)$$

where d_1 and d_{GG} represent the Wilson coefficients for the operators $\mathbf{1}$ and $G^2 = G^{\mu\nu a} G_{\mu\nu}^a$, respectively. The (field-dependent) lowest dimension operator G^2 has dimension four. Its VEV is known as the local gluon condensate and is believed to have a nonzero value of order Λ_{QCD}^4 determined by nonperturbative IR dynamics. This is thought as the origin of the perturbative ambiguity.

The method of our analysis is as follows. We first evaluate $D(Q^2)$ in the large- β_0 approximation [9]. The leading term stems from the diagrams with one gluon propagator in Fig. 1 (diagrams I). We consider insertion of a chain of one-loop fermion self-energies (with n_f flavors) to the gluon propagator. Taking the infinite sum of the chains and replacing $n_f \rightarrow \underline{n_f} - 33/2 = -3\beta_0/2$ gives the diagrams I with $\alpha_s(\mu)$ (μ is the renormalization scale in the $\overline{\text{MS}}$ scheme) replaced by

$$\begin{aligned} \alpha_{\beta_0}(\tau) &\equiv \alpha_s^{1\text{-loop}}(e^{-5/6} \sqrt{\tau}) \\ &= \frac{\alpha_s(\mu)}{1 + \frac{\beta_0 \alpha_s(\mu)}{4\pi} \log(e^{-5/3} \tau / \mu^2)} = \frac{4\pi/\beta_0}{\log(e^{-5/3} \tau / \Lambda_{\text{QCD}}^2)}, \end{aligned} \quad (4)$$

where $\tau = -k^2$ and k denotes the gluon momentum. (We set $n_f = 1$ in the following.) Then loop integrals except for the modulus of the (Euclidean) gluon momentum can be performed, and we obtain the one-dimensional integral expression for the reduced Adler function [9, 10]:

$$\begin{aligned} D_{\beta_0}^{\text{formal}}(Q^2) &= \int d^4p d^4\kappa \mathcal{F}(p, \kappa, Q) \frac{\alpha_{\beta_0}(\kappa^2)}{\kappa^2} \\ &= \int_0^\infty \frac{d\tau}{2\pi\tau} \alpha_{\beta_0}(\tau) \int d^4p d^4\kappa \mathcal{F}(p, \kappa, Q) 2\pi\delta(\tau - \kappa^2) \\ &= \int_0^\infty \frac{d\tau}{2\pi\tau} w_D\left(\frac{\tau}{Q^2}\right) \alpha_{\beta_0}(\tau). \end{aligned} \quad (5)$$

Here, $\alpha_s(\mu)\mathcal{F}(p, \kappa, Q)/\kappa^2$ represents the integrand of the two loop integral expression for the reduced Adler function, and p, κ denote the Euclidean loop momenta ($\tau = \kappa^2 = -k^2$). The above expression is only formal³ due to existence of the pole at $\tau = e^{5/3}\Lambda_{\text{QCD}}^2$ in $\alpha_{\beta_0}(\tau)$ and makes sense only in series expansion in $\alpha_s(\mu)$. In our formalism, we focus on its UV contributions by introducing an IR cut-off to the gluon momentum,

$$D_{\beta_0}(Q^2; \mu_f) \equiv \int_{\mu_f^2}^\infty \frac{d\tau}{2\pi\tau} w_D\left(\frac{\tau}{Q^2}\right) \alpha_{\beta_0}(\tau), \quad (6)$$

where the factorization scale is chosen to satisfy $\Lambda_{\text{QCD}}^2 \ll \mu_f^2 \ll Q^2$. In this definition the integral path does not include the pole and the integral is well-defined. Although eq. (6) generally depends on μ_f , we can extract a μ_f -independent part. Such a part is insensitive to IR physics and can be regarded as a genuine UV contribution.

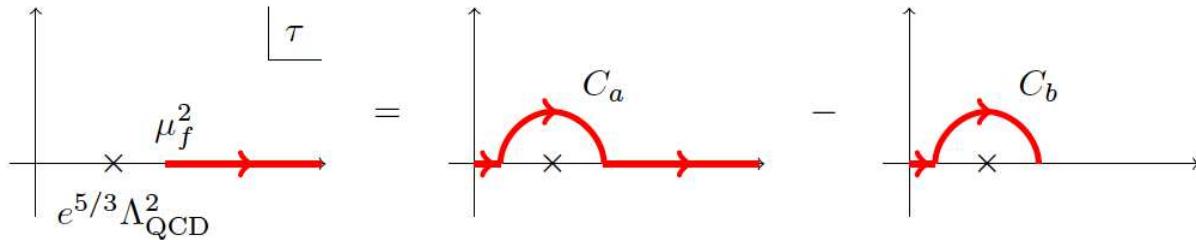
Consider a function $W_D(z)$ which is analytic in the upper-half complex z plane and satisfies

$$2 \text{Im} W_D(x) = w_D(x) \quad (x \in \mathbb{R} \text{ and } x > 0). \quad (7)$$

Then D_{β_0} can be expressed by W_D as

$$D_{\beta_0}(Q^2; \mu_f) = \text{Im} \int_{\mu_f^2}^\infty \frac{d\tau}{\pi\tau} W_D\left(\frac{\tau}{Q^2}\right) \alpha_{\beta_0}(\tau), \quad (8)$$

and the integral path can be decomposed into the difference between C_a and C_b given below.



³ Eq. (5) can be made well-defined and precise by regularization. One prescription used extensively is treating the integral as the principal-value integral plus a contribution from the Landau pole.

The integral along C_a is clearly independent of μ_f . The integral along C_b also includes μ_f -independent part. Since $\mu_f^2 \ll Q^2$ it would be justified to expand W_D about $\tau = 0$ along the path C_b :

$$D_{\beta_0}^{(C_b)}(Q^2; \mu_f) = \sum_n \text{Im} \int_{C_b} \frac{d\tau}{\pi\tau} c_n \left(\frac{\tau}{Q^2} \right)^n \alpha_{\beta_0}(\tau), \quad (9)$$

where $W_D(z) = \sum_n c_n z^n$. For each term, if $c_n \in \mathbb{R}$, the integral can be written as

$$\begin{aligned} & \text{Im} \int_{C_b} \frac{d\tau}{\pi\tau} c_n \left(\frac{\tau}{Q^2} \right)^n \alpha_{\beta_0}(\tau) \\ &= \frac{1}{2\pi i} \left(\int_{C_b} - \int_{C_b^*} \right) \frac{d\tau}{\tau} c_n \left(\frac{\tau}{Q^2} \right)^n \alpha_{\beta_0}(\tau), \end{aligned} \quad (10)$$

since the integrand satisfies the relation $\{f(z)\}^* = f(z^*)$. This reduces to a contour integral surrounding the pole at $\tau = e^{5/3} \Lambda_{\text{QCD}}^2$ and the result is its residue:

$$[\text{eq. (10)}] = -\frac{4\pi c_n}{\beta_0} \left(\frac{e^{5/3} \Lambda_{\text{QCD}}^2}{Q^2} \right)^n. \quad (11)$$

On the other hand, in the case that c_n has a nonzero imaginary part, μ_f -dependence generally remains since the integrand does not satisfy $\{f(z)\}^* = f(z^*)$. In this way μ_f -independent part appears from the integral along C_b depending on whether the expansion coefficient is real or complex.

The analytic function W_D which is related to w_D by eq. (7) can be constructed systematically. If we define

$$W_D(z) = \int_0^\infty \frac{dx}{2\pi} \frac{w_D(x)}{x - z - i0} \quad (z \in \mathbb{C}), \quad (12)$$

it has the desired property. Using w_D computed in [10] we obtain, after appropriate change of conventions,

$$\begin{aligned} W_D(z) = & \frac{N_c C_F}{12\pi} \left[3 + 16z(z+1)H(z) - 14z^2 \log(-z) \right. \\ & + 8z(z+1) \{ -\log(-z) \text{Li}_2(-z) + \text{Li}_3(z) + \text{Li}_3(-z) \} \\ & + 4 \{ 2z^2 + 2z + 1 - 4z(z+1) \log(1+z) \} \text{Li}_2(z) \\ & + 2(7z^2 - 4z - 3) \log(1-z) - 8\zeta_2 z(z+1) \log(1+z) \\ & + 4 \{ z^2 - z(z+1) \log(1+z) \} \log^2(-z) \\ & \left. + 2(4\zeta_2 - 7\zeta_3)z^2 + 2(11 - 7\zeta_3)z \right], \end{aligned} \quad (13)$$

where $N_c = 3$ is the number of colors and $C_F = 4/3$ is the Casimir operator of the fundamental representation; $H(z) = \int_z^1 dx x^{-1} \log(1+x) \log(1-x)$; $\text{Li}_n(z) = \sum_{k=1}^\infty \frac{z^k}{k^n}$ denotes the polylogarithm; $\zeta_k = \zeta(k)$ denotes the Riemann zeta function.⁴

⁴ $H(z)$ can be expressed by the harmonic polylogarithms.

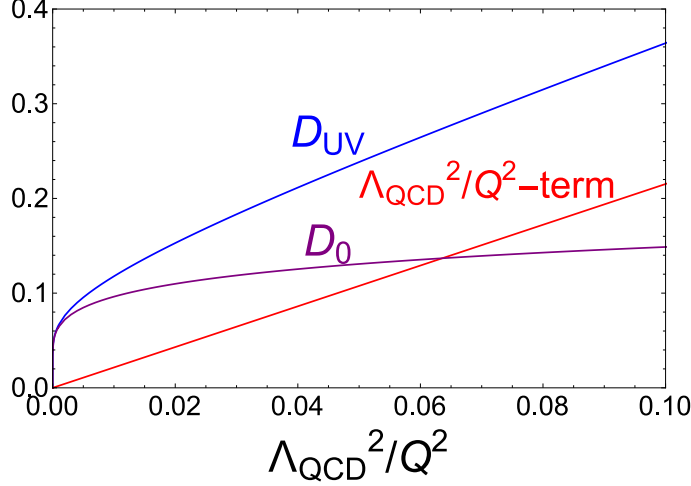


Figure 2: D_{UV} , D_0 and the Λ_{QCD}^2/Q^2 term of eq. (16) as functions of Λ_{QCD}^2/Q^2 .

The expansion of $W_D(z)$ in z reads

$$W_D(z) = N_c C_F \left[\frac{1}{4\pi} + \frac{8 - 6\zeta_3}{3\pi} z + \frac{10 - 12\zeta_3 - 3 \log z + 3i\pi}{6\pi} z^2 + \dots \right]. \quad (14)$$

The first two terms have real expansion coefficients, whereas the third term has a complex coefficient. As a result we obtain

$$D_{\beta_0}(Q^2; \mu_f) = D_{UV}(Q^2) + \mathcal{O}(\mu_f^4/Q^4), \quad (15)$$

with

$$D_{UV}(Q^2) = D_0(Q^2) + \frac{8(4 - 3\zeta_3)e^{5/3} N_c C_F \Lambda_{QCD}^2}{3\beta_0 Q^2}, \quad (16)$$

$$D_0(Q^2) = \frac{N_c C_F}{\beta_0} + \text{Im} \int_{C_a} \frac{d\tau}{\pi\tau} W_D\left(\frac{\tau}{Q^2}\right) \alpha_{\beta_0}(\tau). \quad (17)$$

The Λ_{QCD}^2/Q^2 term with the same coefficient has been obtained in [18]. The asymptotic behaviors of $D_0(Q^2)$ can be calculated analytically, which reads

$$D_0(Q^2) \rightarrow \begin{cases} \frac{N_c C_F}{\beta_0} & \text{as } Q^2 \rightarrow 0 \\ \frac{N_c C_F}{\beta_0} \frac{1}{\log(Q^2/\Lambda_{QCD}^2)} & \text{as } Q^2 \rightarrow \infty \end{cases}. \quad (18)$$

[The behavior as $Q^2 \rightarrow \infty$ is consistent with the renormalization group (RG).] In the intermediate region both asymptotic forms are interpolated smoothly. $D_0(Q^2)$ can be easily computed numerically, which is shown in Fig. 2. D_0 has a logarithmic dependence on Λ_{QCD}/Q , which is milder compared to power dependences. Thus, in this way we generate effectively an expansion in $1/Q^2$, and D_{UV} can be regarded as the leading terms in this expansion. (This expansion is not unique due to the reason discussed below.)

D_{UV} is determined only by UV contributions and μ_f -independent, i.e., insensitive to IR physics. Especially $1/Q^2$ -term is included and this gives a more dominant contribution

than $1/Q^4$ -term at large Q^2 . In addition, Λ_{QCD}^2 , which cannot be expanded in α_s , appears together with $1/Q^2$. μ_f -dependent terms start from the order $1/Q^4$, which is consistent with the fact that $1/Q^4$ -term has IR contributions in OPE.⁵ D_{UV} is plotted in Fig. 2 as a function of $(\Lambda_{\text{QCD}}/Q)^2$, in which linear dependence on $(\Lambda_{\text{QCD}}/Q)^2$ is visible.

In general the correspondence between perturbative calculation and OPE in an effective field theory can be examined using expansion-by-regions of Feynman diagrams [13]. According to such an analysis D_{β_0} coincides with d_1 , since D_{β_0} corresponds to the UV gluon part. It is clear from the construction that D_{β_0} and D_{UV} do not contain IR renormalons (which stem from the region $\tau \sim \Lambda_{\text{QCD}}^2$ [8]). In fact D_{UV} has a well-defined value (up to a scheme dependence discussed below).

One may suspect that the $\Lambda_{\text{QCD}}^2/Q^2$ term is an IR contribution since it stems from the contour C_b close to the IR pole at $\tau = e^{5/3}\Lambda_{\text{QCD}}^2$. One can verify that this is a UV contribution using the expansion-by-regions technique. Combining eqs. (5) and (12), we can write

$$W_D\left(\frac{\tau}{Q^2}\right) = \int d^4p d^4\kappa \frac{\mathcal{F}(p, \kappa, q)}{\kappa^2 - \tau - i0}. \quad (19)$$

We separate the momentum regions of the integral and investigate them individually. We use Q^2 as a hard-scale parameter and $\tau \sim \Lambda_{\text{QCD}}^2$ as a soft-scale parameter. In the region where all of the quarks and gluon have hard-scale momenta, eq. (19) becomes

$$W_D\left(\frac{\tau}{Q^2}\right)\Big|_{\text{all hard}} = \sum_{n=0}^{\infty} \int d^4p d^4\kappa \mathcal{F}(p, \kappa, Q) \frac{\tau^n}{(\kappa^2)^{n+1}}. \quad (20)$$

Note that the function \mathcal{F} does not receive any modification in this region since the soft-scale parameter τ is contained only in the factor $1/(\kappa^2 - \tau)$. The first and the second ($n = 0, 1$) terms of eq. (20) exactly reproduce the first and the second terms of eq. (14), respectively. Therefore we conclude that the $\Lambda_{\text{QCD}}^2/Q^2$ term is a UV contribution.

On the other hand, the imaginary part of c_n in eq. (9) results in μ_f -dependent terms, and the μ_f -dependent terms are identified as IR contributions. The expansion-by-regions analysis shows that the imaginary part of c_n indeed stems from the region where the gluon has a soft-scale momentum. This is because the only source of imaginary part is the integral of $1/(\kappa^2 - \tau)$, namely, once this factor is expanded as in eq. (20), the contribution is explicitly real.

In ref. [18], using the method of massive gluon, terms which are non-analytic in the gluon mass λ are identified as IR contributions, while terms which are power-like in λ as UV contributions. Written in the form eq. (19), it has the same structure as the massive gluon with a negative mass-squared. Hence, the source of the imaginary part can be attributed to the same origin. For example, a non-analytic term $\ln \lambda^2$ generates an imaginary part when we substitute $\lambda^2 = -\tau$ with $\tau > 0$.

⁵ In the static potential the leading μ_f dependence (corresponding to the μ_f^4/Q^4 term in the Adler function) cancels against that of the leading nonperturbative matrix element (non-local gluon condensate) [15, 14]. We expect that a similar cancellation takes place also for the Adler function. To show this explicitly requires computation of the matrix element in a Wilsonian low-energy effective field theory with a hard cut-off.

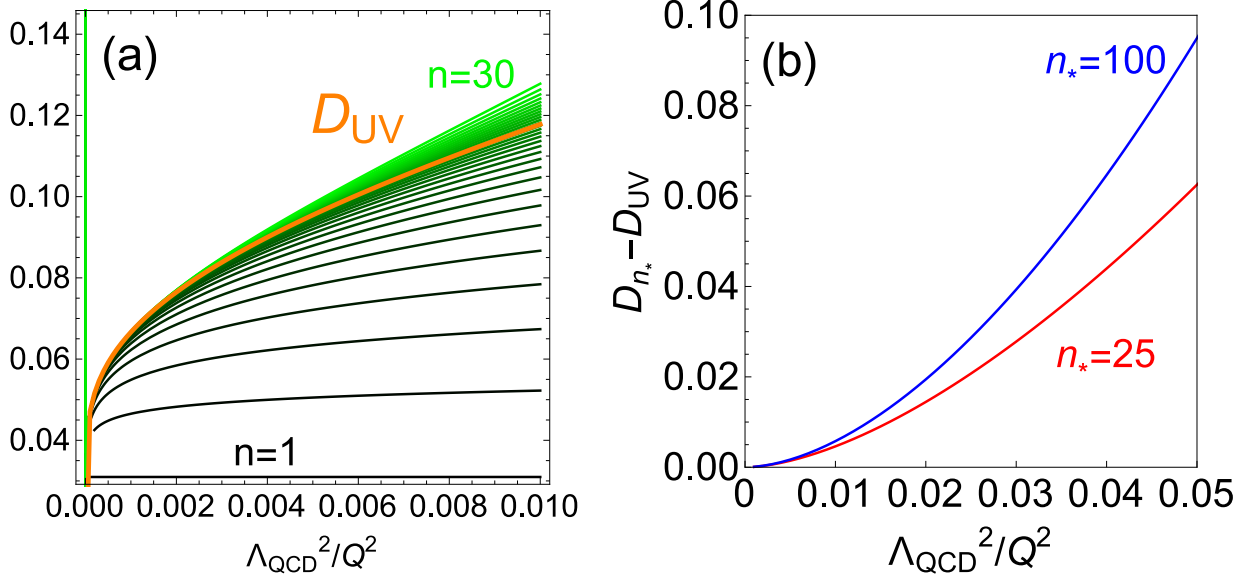


Figure 3: (a) Sum of series expansion of $D_{\beta_0}^{\text{formal}}$ up to $\mathcal{O}(\alpha_s^n)$ and D_{UV} . The input is $\alpha_s(\mu) = 0.097$ (corresponding to $n_* = 25$). (b) $D_{n_*} - D_{\text{UV}}$ vs. $\Lambda_{\text{QCD}}^2/Q^2$, for $n_* = 25$ and 100 .

W_D is not unique because we can add an arbitrary function which takes a real value on the positive real axis preserving the condition (7). One can show⁶ that this changes D_{UV} only at order $(\Lambda_{\text{QCD}}/Q)^4$ or higher, although both D_0 and coefficient of $(\Lambda_{\text{QCD}}/Q)^2$ vary. The reason why the coefficient of $(\Lambda_{\text{QCD}}/Q)^2$ cannot be determined uniquely is due to the existence of the $1/\log Q^2$ singularity in eq. (18) dictated by RG, which prevents Taylor expansion in $1/Q^2$. Dependence of D_{UV} on the choice of W_D may be regarded as a scheme dependence, tied with the non-existence of Taylor expansion.

We can also extract the same D_{UV} by truncating the formal series expansion (5) at order $n_* \approx 8\pi/(\beta_0\alpha_s(\mu))$ (D_{n_*}) and examining the limit $\alpha_s(\mu) \rightarrow 0$, following the analysis method in [11]. As $\alpha_s(\mu) \rightarrow 0$ ($n_* \rightarrow \infty$) the truncated series approaches D_{UV} up to a slowly diverging $\mathcal{O}(\Lambda_{\text{QCD}}^4/Q^4)$ part, i.e., $D_{n_*} - D_{\text{UV}} \sim \log n_* \times \mathcal{O}(\Lambda_{\text{QCD}}^4/Q^4)$ for $n_* \gg 1$. In Fig. 3(a) we compare D_{UV} and the sum of the truncated series up to $\mathcal{O}(\alpha_s^n)$ in the case $\alpha_s(\mu) = 0.097$ (corresponding to $n_* = 25$). We show in Fig. 3(b) the difference $D_{n_*} - D_{\text{UV}}$ for the choices of $\alpha_s(\mu)$ corresponding to $n_* = 25$ and 100 . The difference for each n_* indeed behaves as $\mathcal{O}(\Lambda_{\text{QCD}}^4/Q^4)$, namely it reduces at larger Q^2 .

So far we have analyzed in the large- β_0 approximation. The exact perturbative series is known up to $\mathcal{O}(\alpha_s^4)$ [4]. We compare both results up to this order in Fig. 4 and see qualitatively a good agreement. Hence, our analysis in the large- β_0 approximation looks consistent. Namely, it suggests that at large ($\lesssim n_*$) order of perturbative expansion linear dependence on $(\Lambda_{\text{QCD}}/Q)^2$ would appear; see Fig. 3(a).

⁶ The variation of D_0 and that of c_1 -term, caused by a variation of W_D (δW_D), cancel up to a residual variation of order $(\Lambda_{\text{QCD}}/Q)^4$. Note that $\{\delta W_D(z)\}^* = \delta W_D(z^*)$. Furthermore, one can show that the μ_f -dependent part does not vary.

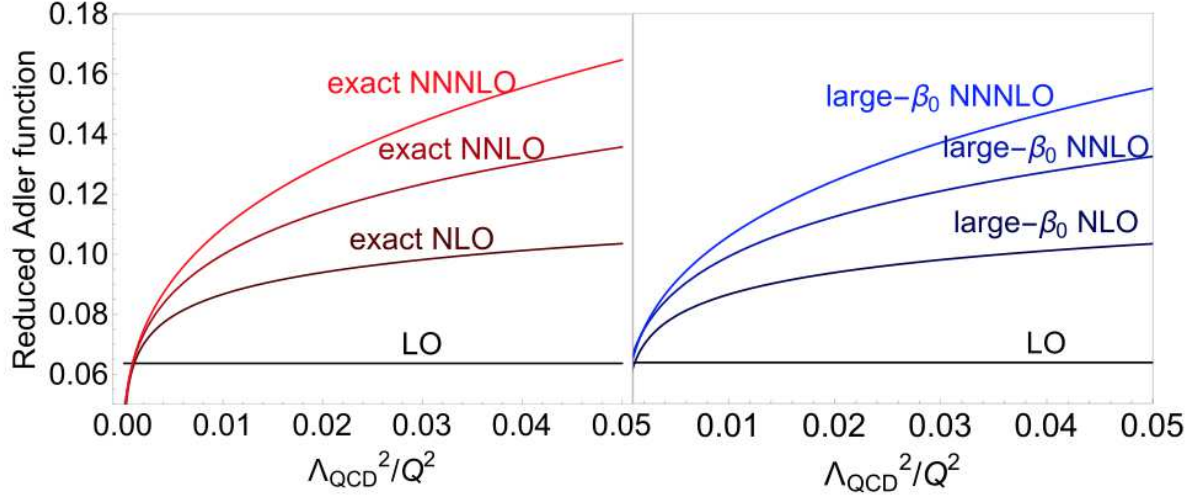


Figure 4: Perturbative series of $D(Q^2)$: exact result for the non-singlet part (left) and large- β_0 approximation (right). $N^k\text{LO}$ line represents the sum of the series up to $\mathcal{O}(\alpha_s^{k+1})$. The input is taken as $\alpha_s(\mu) = 0.2$.

Thus, we have extracted UV contributions to the Adler function, free from IR renormalons, and presented an understanding of the $\Lambda_{\text{QCD}}^2/Q^2$ term included in it. Namely, the leading Wilson coefficient d_1 can be identified with D_{UV} up to order μ_f^4/Q^4 , where D_{UV} is given as a sum of a logarithmic term D_0 and a $\Lambda_{\text{QCD}}^2/Q^2$ term. We showed that the $\Lambda_{\text{QCD}}^2/Q^2$ term is indeed included in large order perturbative series. Note that this power behavior is different from a perturbative uncertainty induced by the UV renormalon located on the negative real axis in the Borel plane. The contribution from UV renormalon is renormalization scale dependent and becomes less important as we raise the order of truncation (n_*) properly. Note also that the separation into the D_0 and $\Lambda_{\text{QCD}}^2/Q^2$ terms is not unique. It is difficult to eliminate this dependence by expansion in $1/Q^2$ due to $1/\log Q^2$ singularity in the leading term. This scheme dependence is *not* a perturbative uncertainty and eventually cancels in $D(Q^2)$. From comparisons in Figs. 2 and 3 we find the scheme choice given by eq. (12) a reasonable one. In this way the $\Lambda_{\text{QCD}}^2/Q^2$ term is intrinsic to the perturbative prediction of the Adler function.

In the case of the static potential a method of systematic improvement (beyond large- β_0 approximation) was devised and applied, which resulted in a better agreement with lattice results [11, 14]. Unfortunately the same method does not work for the Adler function. Nevertheless in principle any improvement in the UV region justifiable in perturbative QCD should be valid since our method depends only on this part.

Finally we remark that the formulation presented in this Letter would be applicable to more general observables, at least to those which depend only on one scale and in the large- β_0 approximation.

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